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Robert J. Elliott

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Department of Statistics and Applied Probability
University of Alberta
Edmonton, Alberta, Canada
T6G 2G1

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Filters With Small Non-Linearities

Robert J. Elliott

Department of Statistics and Applied Probability
University of Alberta
Edmonton, Alberta, Canada T6G 2G1

ABSTRACT

The Kalman filter provides a finite dimensional solution
when the signal and observation processes are linear and
have Gaussian noise. In this paper the effect of a small non-
linearity in the signal is discussed by considering stochastic
flows for the signal and a Girsanov transformation for the
observation. The result can be expressed in terms of Gaus-
sian densities.

1. THE LINEAR FILTER

In this section we first describe the linear Kalman filter. For
simplicity real valued signal and observation processes will
be considered; the vector case can be discussed with more
complicated notation and calculations. ω_t, B_t, $t \geq 0$, are
two independent Brownian motions defined on a probabil-
ity space (Ω, F, P) which has a complete, right continuous
filtration $\{F_t\}$ to which ω and B are adapted. a_t, $t \geq 0$, is
a locally integrable, measurable function, and h_t, $t \geq 0$, is
a function with a locally integrable derivative.

Suppose the SIGNAL is described by the linear equation

$$x_t = x_s + \int_s^t a_u x_u du + \omega_t. \qquad (1.1)$$

Write the solution of (1.1) as $\xi_{s,t}(x_s)$. Supose $\Phi(s,t)$ is the
solution of

$$\frac{d\Phi(s,t)}{dt} = a_t \Phi(s,t)dt, \qquad t \geq s, \qquad (1.2)$$

$$\Phi(s,s) = 1.$$

Clearly, $\Phi(s,t) = \exp\left(\int_s^t a_u du\right)$ and

$$\xi_{s,t}(x_s) = \Phi(s,t)\left\{x_s + \int_s^t \Phi(s,u)^{-1}d\omega_u\right\}. \qquad (1.3)$$

The OBSERVATION process is taken to be of the form

$$y_t = \int_0^t h_s \xi_{0,s}(x_0)ds + B_t. \qquad (1.4)$$

As usual, we shall suppose x_0 is a Gaussian F_0 measurable
random variable independent of ω_t, B_t, $t > 0$.

Write $\{Y_t\}$, $t \geq 0$, for the right continuous complete filtra-
tion generated by the observations and

$$\hat{x}_t(x_s) = E[x_t \mid x_s, Y_t] \qquad \text{for} \quad t \geq s.$$

Then it is known that $\hat{x}_t(x_s)$ is a Gaussian random variable
for $t > s$ and

$$\hat{x}_t(x_s) = x_s + \int_s^t a_u \hat{x}_u(x_s)du$$
$$+ \int_s^t P_{s,u} h_u (dy_u - h_u \hat{x}_u(x_s))du \qquad (1.5)$$

where

$$P_{s,t} = E[x_t^2 \mid x_s, Y_t] - (E[x_t \mid x_s, Y_t])^2$$

satisfies the deterministic equation

$$\frac{dP_{s,t}}{dt} = -h_t^2 P_{s,t}^2 + 2a_t P_{s,t} + 1, \qquad (1.6)$$

$$P_{s,s} = 0.$$

Consequently, $\hat{x}_t(x_s)$ is Gaussian with conditional mean
$\hat{x}_t(x_s)$ and variance $P_{s,t}$.

Writing $\hat{x}_t = E[x_t \mid Y_t]$ we see $\hat{x}_t$ is Gaussian with mean
and variance P_t given by

$$\hat{x}_t = E[x_0] + \int_0^t a_s \hat{x}_s ds + \int_0^t P_s h_s (dy_s - h_s \hat{x}_s ds) \qquad (1.7)$$

$$\frac{dP_t}{dt} = -h_t^2 P_t^2 + 2a_t P_t + 1, \qquad (1.8)$$

$$P_0 = E[x_0^2] - (E[x_0])^2.$$

The equations (1.5) and (1.6), or (1.7) and (1.8) are forms of the Kalmar filter. The inovation processes

$$\beta_t(x_s) = y_t - \int_s^t h_u \hat{x}_u(x_s) du, \qquad t \geq s,$$

$$\beta_t = y_t - \int_0^t h_u \hat{x}_u du, \qquad t \geq 0,$$

are $\{Y_t\}$ Brownian motions. They generate the same filtration as $\{y_t\}$.

The Gaussian measure on R with mean m and variance P will be denoted by $\mu(m, P, dx)$. If g is a Borel measurable function on R we shall write

$$\Gamma(g, m, P) = \int_R g(x) \mu(m, P, dx).$$

If Z_t is an integrable process, $t \geq 0$, $\Pi_t(Z)$ will denote the $\{Y_t\}$-predictable projection of Z, so $\Pi_t(Z) = E[Z_t \mid Y_t]$ a.s. For a function $g(t, x)$ such that

$$|g(t, x)| \leq K(1 + |x|^m)$$

for some $K > 0$, $m > 0$, we shall write

$$\Pi_t(g) = \Pi_t(g(t, x_t)).$$

From [2] we quote the following results:

LEMMA 1.1. a) *Suppose* $0 \leq s \leq t$. *The conditional law of* x_s *given* Y_t *is*

$$\mu(m_s^t, P_s^t, dx)$$

where

$$m_s^t = \hat{x}_t + \frac{P_s}{\gamma_s} \int_s^t \gamma_u h_u d\beta_u \qquad (1.9)$$

$$P_s^t = P_s - \left(\frac{P_s}{\gamma_s}\right)^2 \int_s^t \gamma_u^2 h_u^2 du \qquad (1.10)$$

and γ *is the solution of*

$$\gamma_t = 1 + \int_0^t (a_s - P_s h_s^2) \gamma_s ds \qquad (1.11)$$

so

$$\gamma_t = \exp \int_0^t (a_s - P_s h_s^2) ds.$$

b) *Suppose* $g(t, x)$ *and* $g_x(t, x)$ *are Borel functions satisfying growth conditions as above. Then*

$$\Pi_t\left(\int_0^t g(s, x_s) ds\right) = \int_0^t \Pi_s(g) ds$$
$$+ \int_0^t \Pi_s\left(\int_0^s g_x(u, x_u) \frac{P_u}{\gamma_u} du\right) \gamma_s h_s d\beta_s. \qquad (1.12)$$

From (1.3) we see the map

$$x \to \xi_{s,t}(x)$$

is a diffeomorphism of R and

$$\frac{\partial \xi_{s,t}(x)}{\partial x} = \Phi(s, t).$$

From (1.5) we can write

$$\hat{x}_t(x_s) = \Phi(s, t)\left[x_s + \int_s^t \Phi(s, u)^{-1} P_{s,u} h_u d\beta_u(x_s)\right] \qquad (1.13)$$

and

$$\frac{\partial \hat{x}_t(x_s)}{\partial x_s} = \gamma_{s,t}$$

where

$$\gamma_{s,t} = 1 + \int_s^t (a_u - P_{s,u} h_u^2) \gamma_{s,u} du \qquad (1.14)$$

so

$$\gamma_{s,t} = \exp \int_s^t (a_u - P_{s,u} h_u^2) du. \qquad (1.15)$$

2. NONLINEAR SIGNAL EQUATIONS

For linear signal and observations the Kalman filter provides a finite dimensional solution to the filtering problem. Consider a measurable function $f(t, x)$ on $[0, \infty) \times R$ which is twice differentiable in x and which satisfies the growth condition

$$|f(t, x)| + |f_x(t, x)| \leq K(1 + |x|). \qquad (2.1)$$

Let $\varepsilon > 0$ be a small positive number. Consider a signal process given by the non-linear equation

$$\bar{x}_t = x_0 + \int_0^t (a_s \bar{x}_s + \varepsilon f(s, \bar{x}_s)) ds + \omega_t. \qquad (2.2)$$

Consider the process z defined by

$$z_t = x_0 + \int_0^t \Phi(0, s)^{-1} \varepsilon f(s, \xi_{0,s}(z_s)) ds \qquad (2.3)$$

where $\xi_{0,s}(\cdot)$ is the diffeomorphism defined by (1.1).

LEMMA 2.1. *The process* $\xi_{0,t}(z_t)$ *is the solution of (2.2).*

PROOF. Substituting (2.3) in (1.3) we have

$$\xi_{0,t}(z_t) = \Phi(0, t)\left[x_0 + \int_0^t \Phi(0, s)^{-1} \varepsilon f(s, \xi_{0,s}(z_s)) ds\right.$$
$$\left. + \int_0^t \Phi(0, s)^{-1} d\omega_s\right]. \qquad (2.4)$$

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Differentiating (2.4) in t the result follows.

REMARKS 2.2. Because f satisfies the linear growth condition (2.1) $\bar{x}_t = \xi_{0,t}(z_t)$ has finite moments of all orders.

If Z_t is a process we shall write $Z_t = O(\epsilon^k)$ if

$$\left(E\left(\sup_{s \le t} |Z_t|^p \right) \right)^{1/p} = O(\epsilon^k)$$

for every $p \ge 1$.

NOTATION 2.3. Write

$$\Delta_{0,t} = \Phi(0,t) \int_0^t \Phi(0,s)^{-1} f(s,x_s) ds.$$

Using the mean value theorem we can quickly deduce

PROPOSITION 2.4. $\bar{x}_t - x_t = D_{0,t} = \epsilon \Delta_{0,t} + O(\epsilon^2)$.

REMARKS 2.5. To discuss the effect of the non-linear signal $\bar{x}_t = \xi_{0,t}(z_t)$ on the observations consider the measure $\bar{P}$ defined by

$$\left. \frac{d\bar{P}}{dP} \right|_{F_t} = \Lambda_t^\epsilon$$

where

$$\Lambda_t^\epsilon = \exp\left(\int_0^t h_s D_{0,s} dB_s - \frac{1}{2} \int_0^t h_s^2 D_{0,s}^2 ds \right).$$

Then under $\bar{P}$

$$\bar{B}_t = B_t - \int_0^t h_s D_{0,s} ds$$

is a Brownian motion, i.e.,

$$y_t = \int_0^t h_s \xi_{0,s}(z_s) ds + \bar{B}_t. \tag{2.5}$$

Therefore, under $\bar{P}$ the signal process is $\bar{x}$ and this now influences the observations as in (2.5). The non-linear filtering expression we wish to consider is

$$\bar{E}[\xi_{0,t}(z_t) \mid Y_t].$$

By Baye's theorem this is

$$E[\Lambda_t^\epsilon \xi_{0,t}(z_t) \mid Y_t] \cdot (E[\Lambda_t^\epsilon \mid Y_t])^{-1}.$$

LEMMA 2.6. $\Lambda_t^\epsilon = 1 + \epsilon \int_0^t h_s \Delta_{0,s} dB_s + O(\epsilon^2)$.

PROOF. $\Lambda_t^\epsilon = 1 + \int_0^t \Lambda_s^\epsilon h_s D_{0,s} dB_s$ and the result follows by substituting for Λ_s^ϵ on the right and using Proposition 2.4.

From Proposition 3.3 of Picard [2] we have

LEMMA 2.7. $\Pi_t(\Lambda)^{-1} = 1 - \epsilon \Pi_t\left[\int_0^t h_s \Delta_{0,s} dB_s \right] + O(\epsilon^2)$.

The main result is the following theorem:

THEOREM 2.8. Writing $\bar{x}_t = \xi_{0,t}(z_t)$, $x_t = \xi_{0,t}(x_0)$

$$\bar{E}[\bar{x}_t \mid Y_t] = E[x_t \mid Y_t] + \epsilon E\left[x_t \int_0^t h_s \Delta_{0,s} dB_s \mid Y_t \right]$$

$$+ \epsilon E[\Delta_{0,t} \mid Y_t] - \epsilon E[x_t \mid Y_t] E\left[\int_0^t h_s \Delta_{0,s} dB_s \mid Y_t \right]$$

$$+ O(\epsilon^2). \tag{2.6}$$

PROOF.

$$\bar{E}[\bar{x}_t \mid Y_t] = E[\Lambda_t^\epsilon \bar{x}_t \mid Y_t] \cdot E[\Lambda_t^\epsilon \mid Y_t]^{-1}$$

$$= E[\Lambda_t^\epsilon (x_t + \epsilon \Delta_{0,t}) \mid Y_t]$$

$$\times E\left[\left(1 - \epsilon \int_0^t h_s \Delta_{0,s} dB_s \right) \mid Y_t \right] + O(\epsilon^2)$$

$$= E\left[\left(1 + \epsilon \int_0^t h_s \Delta_{0,s} dB_s \right)(x_t + \epsilon \Delta_{0,t}) \mid Y_t \right]$$

$$\times \left[1 - \epsilon E\left[\int_0^t h_s \Delta_{0,s} dB_s \mid Y_t \right] \right] + O(\epsilon^2)$$

by Proposition 2.3 and Lemma 2.7.

REMARKS 2.9. These expectations are all expressible in terms of Gaussian measures because they are all expectations of functions of the original linear process x_t under the original measure P. For example, $E[x_t \mid Y_t] = \hat{x}_t$ is given by the Kalmar filter. The remaining terms in (2.6) can be expressed in a recursive way; proofs can be found in [1]. For example, we have

LEMMA 2.10.

$$E[\Delta_{0,t} \mid Y_t] = \Phi(0,t)\left[\int_0^t \Phi(0,s)^{-1} \Pi_s(f(s,x_s)) ds \right.$$

$$+ \left. \int_0^t \Pi_s\left\{ \int_0^s f_x(u,x_u) P_u \gamma_u^{-1} du \right\} \Phi(0,s)^{-1} h_s \gamma_s d\beta_s \right].$$

3. CONCLUSION

As in the paper of Picard [2] the first two terms in an expansion of the conditional mean in powers of ϵ have been determined. These coefficients have been expressed explicitly in terms of Gaussian measures by using stochastic flows.

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[1] R.J. Elliott, "Filtering with a small non-linear term in the signal," Technical report, Department of Statistics and Applied Probability, University of Alberta. Submitted, 1988.

[2] J. Picard, "A filtering problem with a small nonlinear term." Stochastics, 18, pp. 313–341, 1986.